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The Transmission of Curved Neutron Guides with Non-perfect Reflectivity

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Abstract

Approximate analytic expressions for determining the transmission of neutron benders which take into account reflectivity losses using the mean number of reflections agree well with computer simulation results which track the neutron trajectories directly, provided that the reflectivity is high. The exact calculation which can be performed using exponential integral functions for any value of the reflectivity is presented. Ray-tracing techniques for determining the transmission and number of reflections for non-perfect reflectivity can therefore be replaced by direct calculation.

1. Introduction

Beam benders and curved guides can be used on neutron guide systems for steering the beam over short distances (Sutherland & Wroe, 1975) and for increasing the number of experimental facilities which can be placed at end positions (Schirmer, Heitjans, Faber & Samuel, 1990); they can also be used as devices on neutron optical systems (Rekveldt & Kraan, 1983; Rekveldt, Verkerk & van Well, 1988; de Haan, Kraan & van Well, 1990). Ray-tracing simulation results of neutron trajectories through a beam bender have been shown (Schirmer & Mildner, 1991) to be in agreement with directly calculated transmission curves taking into account the average number of reflections made within the beam bender when the reflectivity R is very close to unity. The increasing deviation of the calculation from the results of the simulation at higher wavelengths is accounted for by the difference in the method of averaging. The agreement can be improved when the averaging takes into account only those neutrons which are successfully transmitted, so that the mean number of reflections decreases for $R < 1$ relative to that for perfect reflectivity ($R = 1$). This approximation is useful for bender designs and can replace Monte Carlo computations. However, the agreement decreases as the reflectivity deviates significantly from unity because the expansion of R^n (where n is the number of reflections) is an alternating series. At the longer wavelengths the number of reflections increases and the series diverges, causing poor agreement with the simulation.

We consider the exact calculation of the transmission of the beam bender when the reflectivity is not perfect. We also determine the average number of reflections for neutrons of a particular wavelength. The assumption is that the entrance of the beam bender is completely and uniformly illuminated by the beam so that all possible successful trajectories are analyzed, as in the simulation.

2. Perfect reflectivity

The transmission properties of the beam bender or curved guide (Maier-Leibnitz & Springer, 1963) are determined by a characteristic angle ψ_c which is defined by the dimensions of the bender as $\psi_c = (2H/\rho)^{1/2}$, where H is the width of the slots and ρ is the radius of curvature of the bender, provided that $H \ll \rho$. If the height of the bender is much greater than the width of the slots, we may consider the transmission as a one-dimensional problem. The transmission function of the beam bender as a function of wavelength λ depends on the characteristic wavelength λ_c which is determined by $\lambda_c = \psi_c/\gamma_c$, where γ_c is the critical angle of reflection per unit wavelength for the reflecting surface.

The transmission $T(\lambda)$ of the bender without reflectivity losses rises from zero at very short wavelengths to 2/3 at the characteristic wavelength λ_c, and goes asymptotically to unity at the longest wavelengths; that is,

$$T(\lambda) = 2/3(\lambda/\lambda_c)^2 \qquad \lambda \leq \lambda_c$$
$$T(\lambda) = 2/3(\lambda_c/\lambda)\{(\lambda/\lambda_c)^3 - [(\lambda/\lambda_c)^2 - 1]^{3/2}\} \qquad \lambda > \lambda_c. \tag{1}$$

For $\lambda < \lambda_c$, only garland reflections can occur, whereas zigzag reflections can occur for $\lambda > \lambda_c$ and the transmitted spectrum is harder at the concave edge of the bender relative to the convex edge.

The total number of reflections $n(\chi)$ for both garland and zigzag trajectories which undergo reflection at a particular grazing angle χ at the outer (concave) surface of the guide is given (Mildner, 1990*a*) by

$$n(\chi) = L/2\rho\chi = (L/L_c)\psi_c/\chi \qquad \chi \leq \psi_c$$

and (2)

$$n(\chi) = L/\rho[\chi - (\chi^2 - \psi_c^2)^{1/2}]^{-1} = 2(L/L_c)\psi_c[\chi - (\chi^2 - \psi_c^2)^{1/2}]^{-1} \qquad \chi > \psi_c.$$

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These expressions are valid if the bender length L is at least as long as the length of direct sight $L_c = (8H\rho)^{1/2}$. The average number of reflections at a wavelength λ is obtained by the integral of $n(\chi)$ over all grazing angles within the distribution $m(\chi)$ from zero up to the critical angle θ_c corresponding to the wavelength λ:

$$\langle n \rangle = \int_0^{\theta_c} n(\chi) m(\chi)\, d\chi \Big/ \int_0^{\theta_c} m(\chi)\, d\chi. \qquad (3)$$

The distribution $m(\chi)$ of grazing angles for the bender both for garland and for zigzag trajectories is given by

$$m(\chi) = 4H\chi^2/\psi_c^2 \qquad \chi \le \psi_c$$

and (4)

$$m(\chi) = 4H[\chi^2 - \chi(\chi^2 - \psi_c^2)^{1/2}]/\psi_c^2 \qquad \chi > \psi_c.$$

Then the average number of reflections for such trajectories as a function of wavelength is given (Mildner, 1990*b*) by

$$\langle n \rangle = (3/2)(L/L_c)(\lambda_c/\lambda) \qquad \lambda \le \lambda_c,$$

all of which have garland reflections, and (5)

$$\langle n \rangle = (3/2)(L/L_c)[2(\lambda/\lambda_c)^2 - 1] \times \{(\lambda/\lambda_c)^3 - [(\lambda/\lambda_c)^2 - 1]^{3/2}\}^{-1} \qquad \lambda > \lambda_c,$$

for which some of the trajectories have garland reflections and others have zigzag reflections.

3. Non-perfect reflectivity

If the reflectivity of the surfaces of the bender is not unity, the transmission for a particular trajectory must be modified by a factor R^n, where n is the number of reflections for that trajectory. In the computer simulation all possible successful trajectories at a particular wavelength must be sampled randomly, so that the simulation performs an average $\langle R^n \rangle$, where $\langle \ldots \rangle$ indicates an average over all trajectories at a given wavelength.

The exact expression for $\langle R^n \rangle$ is given by

$$\langle R^n \rangle = \int_0^{\theta_c} R^{n(\chi)} m(\chi)\, d\chi \Big/ \int_0^{\theta_c} m(\chi)\, d\chi, \qquad (6)$$

where the number of reflections $n(\chi)$ as a function of grazing angle χ is given by (2) and the distribution $m(\chi)$ of grazing angles by (4). The distribution $m(\chi)$, illustrated in Fig. 1, is always finite for $\chi > 0$ and is zero for $\chi = 0$. The number of reflections $n(\chi)$, illustrated in Fig. 2, is always finite for finite χ and is infinite for $\chi = 0$ and $\pi/2$. Since the value of the reflectivity R is always less than unity, the integrand of the numerator is well behaved. With the transmission [(1)] given by

$$\int_0^{\theta_c} m(\chi)\, d\chi \Big/ 2\int_0^{\theta_c} d\chi,$$

we can evaluate (6) to give

$$\langle R^n \rangle = 3(\lambda_c/\lambda)^3 |\log R|^3 \int_{(\lambda_c/\lambda)|\log R|}^{\infty} e^{-v} v^{-4}\, dv \qquad \lambda \le \lambda_c$$

and (7)

$$\langle R^n \rangle = \left[\int_0^{\psi_c} R^{n(\chi)} m(\chi)\, d\chi + \int_{\psi_c}^{\theta_c} R^{n(\chi)} m(\chi)\, d\chi\right] \Big/ \int_0^{\theta_c} m(\chi)\, d\chi$$

$$= \{3\lambda_c^3 |\log R|^3 / [\lambda^3 - (\lambda^2 - \lambda_c^2)^{3/2}]\} \times \left\{\int_{|\log R|}^{\infty} e^{-v} v^{-4}\, dv - 2\int_{2|\log R|}^{2x^*} e^{-v} v^{-4}\, dv + [8|\log R|^4]^{-1} \int_{2|\log R|}^{2x^*} e^{-v}\, dv\right\} \qquad \lambda > \lambda_c,$$

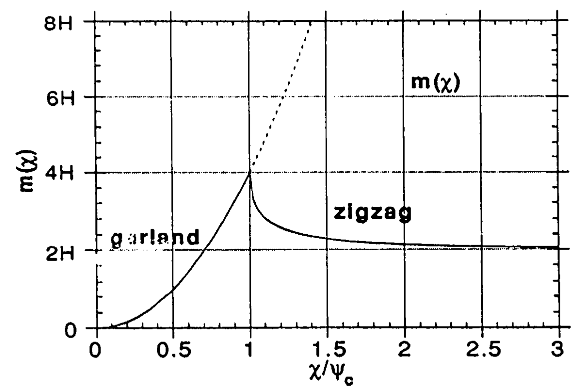

Fig. 1. The distribution $m(\chi)$ of glancing angles χ at the outer surface when a curved guide is fully illuminated in angle and space by neutron trajectories. Glancing angles which correspond to garland reflections for $\chi < \psi_c$ have a distribution $m(\chi) = 4H\chi^2/\psi_c^2$ and to zigzag reflections for $\chi > \psi_c$ [with a glancing angle $(\chi^2 - \psi_c^2)^{1/2}$ at the inner (convex) surface] have a distribution $m(\chi) = 4H[\chi^2 - \chi(\chi^2 - \psi_c^2)^{1/2}]/\psi_c$.

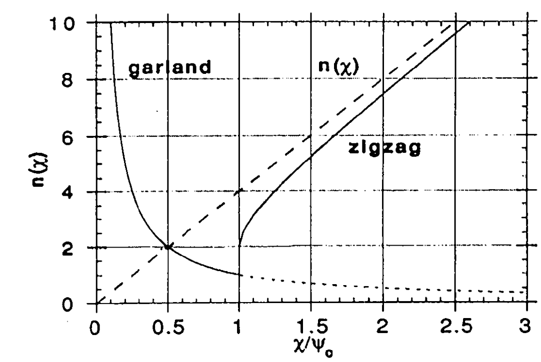

Fig. 2. The total number of reflections $n(\chi)$ as a function of glancing angle χ at the outer surface when a curved guide of length $L_c = (8H\rho)^{1/2}$ is fully illuminated. Trajectories which have garland reflections have $n(\chi) = \psi_c/\chi$, and those with zigzag reflections have $n(\chi) = 2\psi_c/[\chi - (\chi^2 - \psi_c^2)^{1/2}]$. The discontinuity occurs at $\chi = \psi_c$ because zigzag trajectories which have a glancing angle χ at the outer surface also have a reflection with a glancing angle $(\chi^2 - \psi_c^2)^{1/2}$ at the inner surface.

where $x^* = |\log R|\{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}$. Note that $x^* \to \infty$ as $\lambda \to \infty$. Equation (7) is only valid because $R < 1$ and therefore $\log R < 0$. The result may be expressed in terms of the exponential integral functions of the form

$$E_n(\xi) = \int_1^\infty e^{-\xi t} t^{-n}\, dt$$

by

$$\langle R^n \rangle = 3E_4(y) \qquad \lambda \le \lambda_c,$$

where $y = (\lambda_c/\lambda)|\log R|$, and by (8)

$$\langle R^n \rangle = 3\{E_4(x) - [E_4(2x) - (x^*/x)^{-3}E_4(2x^*)]/4 + [E_0(2x) - (x^*/x)E_0(2x^*)]/4\} \times \{(\lambda/\lambda_c)^3 - [(\lambda/\lambda_c)^2 - 1]^{3/2}\}^{-1} \qquad \lambda > \lambda_c,$$

where $x = |\log R|$ and

$$x^* = |\log R|\{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}.$$

If we now use (1), the transmission through the bender of length L_c including reflectivity losses is given by

$$T(\lambda) = 2(\lambda/\lambda_c)^2 E_4(y) \qquad \lambda \le \lambda_c,$$

where $y = (\lambda_c/\lambda)|\log R|$, and (9)

$$T(\lambda) = 2(\lambda_c/\lambda)\{E_4(x) - [E_4(2x) - (x^*/x)^{-3}E_4(2x^*)]/4 + [E_0(2x) - (x^*/x)E_0(2x^*)]/4\} \qquad \lambda > \lambda_c,$$

where $x = |\log R|$ and

$$x^* = |\log R|\{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}.$$

This is shown in Fig. 3 for various values of reflectivity. If the bender has a length L, then the values of the functions y, x and x^* should be multiplied by a factor (L/L_c). Obviously, increasing the length of the bender decreases the transmission for a given wavelength. This acts in the same way as decreasing the reflectivity or increasing the value of $|\log R|$. For good reflectivity of the bender surface, the maximum in the transmission function occurs well above the characteristic wavelength. As the reflectivity becomes poorer and/or the length of the bender increases, the position of maximum transmission moves to lower wavelengths. However, this maximum can never occur below the characteristic wavelength since the condition is $\exp(-y) = E_4(y)$, where $y = (\lambda_c/\lambda)|\log R|$, which has a solution at $y = \infty$ only; in fact, the second differential of $T(\lambda)$ remains positive for $\lambda < \lambda_c$, and, for garland reflections, not only does the transmission for perfect reflectivity increase with wavelength but the number of reflections decreases monotonically. However, at $\lambda = \lambda_c$, the number of reflections abruptly increases. Fig. 4 shows the value of the maximum transmission as a function of reflectivity for a bender of length L_c and Fig. 5 shows the position in wavelength of this maximum. For perfect reflectivity, the transmission tends towards unity at very long wavelengths (that is, the bender acts as a straight guide, see Appendix), whereas for poor reflectivities at very long wavelengths the transmission falls off as $(\lambda|\log R|)^{-1}$, and the maximum in the transmission decreases towards zero and approaches $\lambda = \lambda_c$ asymptotically.

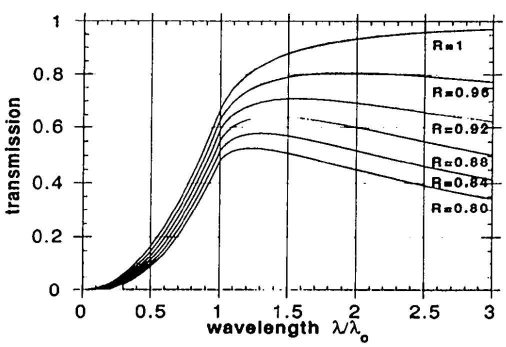

Fig. 3. The transmission through a neutron bender of length $L_c = (8H\rho)^{1/2}$ as a function of wavelength λ for various values of the reflectivity R of the surface.

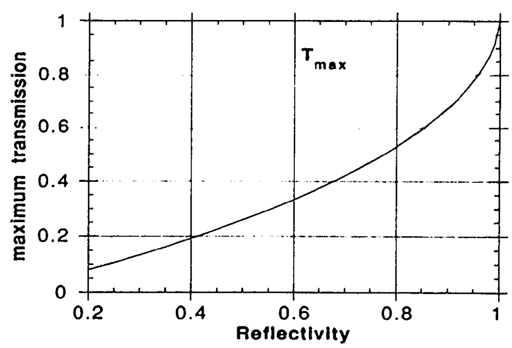

Fig. 4. The value of the maximum transmission for a bender of length L_c as a function of reflectivity R.

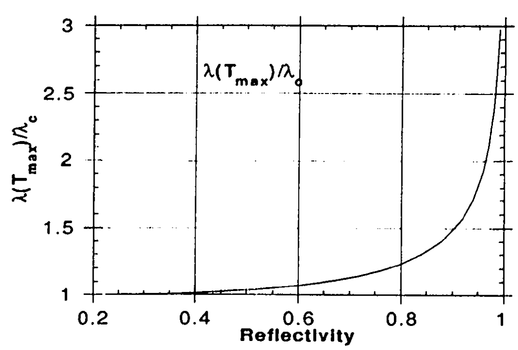

Fig. 5. The position in wavelength of the maximum transmission for a bender of length L_c as a function of reflectivity R.

Also for the mean number of reflections we have

$$\langle n \rangle_R = \int_0^{\theta_c} R^{n(\chi)} n(\chi) m(\chi)\, \mathrm{d}\chi \Big/ \int_0^{\theta_c} R^{n(\chi)} m(\chi)\, \mathrm{d}\chi, \quad (10)$$

which may also be written as

$$\langle n \rangle_R = |\mathrm{d}(\log \langle R^n \rangle)/\mathrm{d}(\log R)|.$$

This is evaluated to give

$$\langle n \rangle_R = (\lambda_c/\lambda) E_3(y)/E_4(y) \qquad \lambda \le \lambda_c,$$

where $y = (\lambda_c/\lambda)|\log R|$, and (11)

$$\langle n \rangle_R = \{E_3(x) - [E_3(2x) - (x^*/x)^{-2} E_3(2x^*)]/2 + [E_{-1}(2x) - (x^*/x)^2 E_{-1}(2x^*)]/2\} \times \{E_4(x) - [E_4(2x) - (x^*/x)^{-3} E_4(2x^*)]/4 + [E_0(2x) - (x^*/x) E_0(2x^*)]/4\}^{-1} \qquad \lambda > \lambda_c,$$

where $x = |\log R|$ and

$$x^* = |\log R| \{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}.$$

This is shown in Fig. 6 for various values of reflectivity. The value of the minimum number of reflections $\langle n \rangle_{R\,\mathrm{min}}$, which always occurs at $\lambda = \lambda_c$, shows a steady decrease from 3/2 at $R = 1$ for $L = L_c$. This is illustrated in Fig. 7 as a function of reflectivity R and for $L = L_c$. Of course, these values cannot be simply multiplied by a factor (L/L_c) for an arbitrary bender length L for values of $R \neq 1$; the values of the functions y, x and x^* should also be multiplied by a factor (L/L_c). In the limit of long wavelengths, $x^* \to \infty$ and $E_n(2x^*) \to 0$, so that the asymptotic value of $\langle n \rangle_R$ for large λ is given by

$$\langle n \rangle_{R\ \mathrm{asympt}} = \frac{E_3(x) - E_3(2x)/2 + E_{-1}(2x)/2}{E_4(x) - E_4(2x)/4 + E_0(2x)/4}. \quad (12)$$

Values of this asymptotic limit as a function of reflectivity R for a bender of length L_c are shown in Fig. 8. These values are consistent with that, $|\log R|^{-1}$, derived in the Appendix for the long straight guide at long wavelengths. Obviously, for perfect reflectivity ($R = 1$), the asymptotic limit is infinite and $\langle n \rangle_R$ approaches $2(\lambda/\lambda_c)$ for $\lambda \gg \lambda_c$.

Similarly, for the mean square number of reflections, we have

$$\langle n^2 \rangle_R = \int_0^{\theta_c} R^{n(\chi)} n(\chi)^2 m(\chi)\, \mathrm{d}\chi \Big/ \int_0^{\theta_c} R^{n(\chi)} m(\chi)\, \mathrm{d}\chi, \quad (13)$$

which may also be written as

$$\langle n^2 \rangle_R = \mathrm{d}^2(\log \langle R^n \rangle)/\mathrm{d}(\log R)^2.$$

This is evaluated to give

$$\langle n^2 \rangle_R = (\lambda_c/\lambda)^2 E_2(y)/E_4(y) \qquad \lambda \le \lambda_c,$$

where $y = (\lambda_c/\lambda)|\log R|$, and (14)

$$\langle n^2 \rangle_R = \{E_2(x) - [E_2(2x) - (x^*/x)^{-1} E_2(2x^*)] + [E_{-2}(2x) - (x^*/x)^3 E_{-2}(2x^*)]\} \times \{E_4(x) - [E_4(2x) - (x^*/x)^{-3} E_4(2x^*)]/4 + [E_0(2x) - (x^*/x) E_0(2x^*)]/4\}^{-1} \qquad \lambda > \lambda_c,$$

where $x = |\log R|$ and

$$x^* = |\log R| \{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}.$$

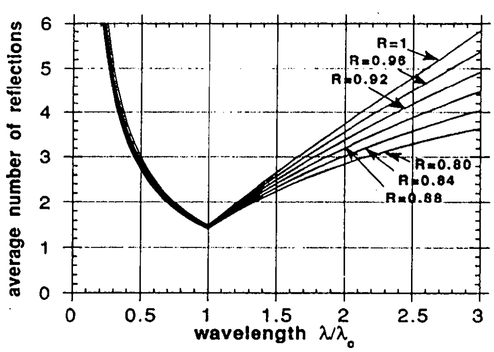

Fig. 6. The average number of reflections for trajectories which pass through a neutron bender of length $L_c = (8H\rho)^{1/2}$ as a function of wavelength λ for various values of the reflectivity R of the surface.

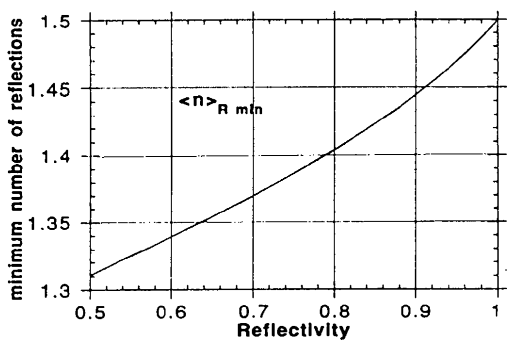

Fig. 7. The minimum number of reflections $\langle n \rangle_{R\,\mathrm{min}}$ at a wavelength $\lambda = \lambda_c$ for a bender of length L_c as a function of reflectivity R.

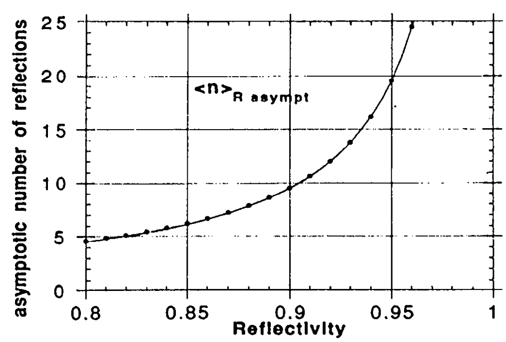

Fig. 8. The asymptotic number of reflections $\langle n \rangle_{R\,\mathrm{asympt}}$ for a bender of length L_c as a function of reflectivity R. The line shown is that of $|\log R|^{-1}$ derived for the straight guide at very long wavelengths.

In the limit of long wavelengths, $x^* \to \infty$ and $E_n(2x^*) \to 0$, so that the asymptotic value of $\langle n^2 \rangle_R$ for large λ is given by

$$\langle n^2 \rangle_{R\,\text{asympt}} = \frac{E_2(x) - E_2(2x) + E_{-2}(2x)}{E_4(x) - E_4(2x)/4 + E_0(2x)/4} \qquad \lambda > \lambda_c. \tag{15}$$

By analogy to the long straight guide at long wavelengths the asymptotic value of $\langle n^2 \rangle_R$ is $2|\log R|^{-2}$. Hence $\langle n^2 \rangle_R - \langle n \rangle_R^2$ approaches $|\log R|^{-2}$ asymptotically.

Schirmer & Mildner (1991) have shown for values of R close to unity that $\langle R^n \rangle$ may be approximated by $R^{\langle n \rangle_R}$, where $\langle n \rangle_R$ is given by

$$\langle n \rangle_R = \langle n \rangle - (1 - R)(\langle n^2 \rangle - \langle n \rangle^2). \tag{16}$$

This result may be found by taking the first-order term in a Taylor-series expansion of $\langle n \rangle_R$ about $\langle n \rangle$ using (10) and (13). Since this result does not involve special functions, it can often be more useful for calculations involving short benders, with a small number of reflections, and with high reflectivity.

4. Different refractive indices

The grazing angle χ at the outer (concave) surface for zigzag trajectory is always greater than the grazing angle χ' at the inner (convex) surface such that

$$\chi'^2 = \chi^2 - \psi_c^2. \tag{17}$$

Hence the surface coatings of the curved guide can have different refractive indices (Schärpf, 1989; de Haan, Kraan & van Well, 1990); that is, the inner surface may have a smaller value of the critical angle per unit wavelength γ_c' than that, γ_c, of the outer (concave) surface, so that $\gamma_c'/\gamma_c = \mu < 1$. For example, the beam bender on the LOQ spectrometer (Heenan *et al.*, 1992) at the ISIS facility uses glass vanes with a supermirror coating only on the concave surface. Provided that the reflection coefficients of the two surfaces are the same, this has no effect on the transmission of neutrons for wavelengths less than some value λ_μ, which is determined by that trajectory for which the grazing angles at each surface are equal to their critical angles at that wavelength; that is,

$$\mu = \frac{\gamma_c' \lambda_\mu}{\gamma_c \lambda_\mu} = \frac{\theta_c'}{\theta_c} = \frac{(\theta_c^2 - \psi_c^2)^{1/2}}{\theta_c} = \frac{(\lambda_\mu^2 - \lambda_c^2)^{1/2}}{\lambda_\mu}. \tag{18}$$

The value of μ is related to n and n', the refractive indices for the concave and convex surface coatings respectively, by $\mu^2 = (1 - n')/(1 - n)$.

Hence this wavelength λ_μ is given by

$$\lambda_\mu = \lambda_c(1 - \mu^2)^{-1/2}. \tag{19}$$

For wavelengths λ greater than λ_μ, the extra acceptance in phase space beyond that for $\lambda = \lambda_\mu$ is determined by the critical angle of the inner (convex) surface and not the outer (concave) surface, because θ_c' is now less than $(\theta_c^2 - \psi_c^2)^{1/2}$. The transmission equation (1) is valid only for the range $\lambda \le \lambda_\mu$ and the transmission for wavelengths greater than λ_μ must be modified. The critical angle $\theta_c' = \gamma_c'\lambda = \mu\gamma_c\lambda = \mu\theta_c$ at the convex surface defines a limiting grazing angle $(\theta_c'^2 + \psi_c^2)^{1/2}$ at the concave surface. In terms of wavelength, the transmission of the curved guide for $\lambda > \lambda_\mu$ is given by (de Haan, Kraan & van Well, 1990)

$$T(\lambda) = 2/3[(\mu^2\lambda^2 + \lambda_c^2)^{3/2} - \mu^3\lambda^3]/\lambda\lambda_c^2$$
$$\lambda > \lambda_\mu = \lambda_c(1 - \mu^2)^{-1/2}. \tag{20}$$

This expression goes to a finite value of μ for infinite λ. This is expected since at long wavelengths the acceptance is determined by the critical angle θ_c' at the convex surface, whereas the input divergence is given by θ_c, so that the transmission varies as θ_c'/θ_c, which by (18) is equal to μ.

If the reflectivities of the two surfaces are different from each other and non-perfect, the exact expression for $\langle R^n \rangle$ for $\lambda > \lambda_c$ is different from (7). Let R and R' be the reflectivities of the concave and convex surfaces respectively. Equation (6) for $\langle R^n \rangle$ and $\lambda > \lambda_c$ should be replaced by

$$\langle R^n \rangle = \left[\int_0^{\psi_c} R^{n(\chi)} m(\chi)\,d\chi + \int_{\psi_c}^{\theta_c^*} R^{n'(\chi)} R'^{n'(\chi)} m(\chi)\,d\chi\right] \times \left[\int_0^{\theta_c} m(\chi)\,d\chi\right]^{-1} \qquad \lambda > \lambda_c, \tag{21}$$

where $n'(\chi) = \psi_c[\chi - (\chi^2 - \psi_c^2)^{1/2}]^{-1}$ is the number of zigzag reflections at each surface. The value of θ_c^* is θ_c for $\lambda_c < \lambda < \lambda_\mu$ and is $(\mu^2\theta_c^2 + \psi_c^2)^{1/2}$ for $\lambda > \lambda_\mu$. We can evaluate (21) to give

$$\langle R^n \rangle = \{3\lambda_c^3/[\lambda^3 - (\lambda^2 - \lambda_c^2)^{3/2}]\} \times \left\{|\log R|^3 \int_{|\log R|}^{\infty} e^{-v} v^{-4}\,dv - \tfrac{1}{4}|\log RR'|^3 \int_{2|\log R|}^{2x^*} e^{-v} v^{-4}\,dv + \tfrac{1}{4}|\log RR'|^{-1} \int_{|\log RR'|}^{2x^*} e^{-v}\,dv\right\} \qquad \lambda > \lambda_c, \tag{22}$$

where $2x^* = |\log RR'|\{(\lambda/\lambda_c) + [(\lambda/\lambda_c)^2 - 1]^{1/2}\}$ for $\lambda_c < \lambda < \lambda_\mu$ [that is, exactly (7) where $2|\log R|$ is replaced by $|\log RR'|$] and $2x^* = |\log RR'|[(\mu^2\lambda^2 + \lambda_c^2)^{1/2} + \mu\lambda]/\lambda_c$ for $\lambda > \lambda_\mu$.

The above result may be expressed in terms of the exponential integral functions by

$$\langle R^n \rangle = 3\{E_4(x) - [E_4(2x') - (x^*/x)^{-3}E_4(2x^*)]/4 + [E_0(2x') - (x^*/x)E_0(2x^*)]/4\} \times \{(\lambda/\lambda_c)^3 - [(\lambda/\lambda_c)^2 - 1]^{3/2}\}^{-1}, \tag{23}$$

where $x = |\log R|$ and $2x' = |\log RR'|$. Hence the transmission through a bender of length L_c including different reflectivity losses from the two surfaces is given by

$$T(\lambda) = 2(\lambda_c/\lambda)\{E_4(x) - [E_4(2x') - (x^*/x)^{-3}E_4(2x^*)]/4 + [E_0(2x') - (x^*/x)E_0(2x^*)]/4\} \qquad \lambda > \lambda_c. \tag{24}$$

5. Concluding remarks

We have shown that the transmission of a neutron bender or curved guide with any value of reflectivity can be determined exactly using exponential integral functions. We assume that the entrance of the bender is illuminated fully and uniformly both in position and in angle. The functional form differs above and below the characteristic wavelength of the guide, and we have determined various properties of the transmission as a function of wavelength. The mean number and the mean-square number of reflections have also been determined for any value of reflectivity in terms of exponential integral functions. These expressions become the same as those for the straight guide in the limit of large wavelengths. From the results, a simple approximate expression for the transmission can be obtained which has been shown useful for short benders with high reflectivity. The calculations have also been applied to the transmission for a beam bender which has different coatings on the inner and outer surfaces of the guide. In addition, a bender with magnetic reflecting surfaces for which γ_c has two values dependent on the direction of the neutron spin relative to the magnetic induction vector enables the device to transmit neutrons with one spin state and not the other by having different reflection properties for each state (Hayter, Penfold & Williams, 1978).

APPENDIX

We derive similar results for the straight guide. The transmission properties of the straight guide are determined by the guide divergence angle $\psi_0 = H/L$, where H is the distance between the inner walls of the guide and L is the length of the guide. The distribution $m(\chi)$ of grazing angles for the straight guide for $\theta_c \leq \psi_0$ is given by

$$\begin{aligned} m(\chi) &= 2H & \chi &\leq \theta_c \\ m(\chi) &= 2H(\psi_0 - \chi)/\psi_0 & \chi &> \theta_c \end{aligned} \tag{A1}$$

and for $\theta_c > \psi_0$ by

$$m(\chi) = 2H \qquad \text{all } \chi.$$

All trajectories are necessarily zigzag and the number of reflections as a function of grazing angle is given by $n(\chi) = \chi/\psi_0$. For perfect reflectivity, the transmission is unity and the average number of reflections at a wavelength λ is obtained by the integral of $n(\chi)$ over all grazing angles within the distribution $m(\chi)$ from zero up to the critical angle θ_c corresponding to the wavelength λ ($= \theta_c/\gamma_c$). That is,

$$\langle n \rangle = \int_0^{\theta_c} n(\chi)m(\chi)\,d\chi \Big/ \int_0^{\theta_c} m(\chi)\,d\chi. \tag{A2}$$

The mean number of reflections is therefore given by

$$\begin{aligned} \langle n \rangle &= \lambda^2/(\lambda^2 + \lambda_0^2) & \lambda &< \lambda_0 \\ \langle n \rangle &= \lambda/2\lambda_0, & \lambda &\geq \lambda_0 \end{aligned} \tag{A3}$$

where $\lambda_0 = \psi_0/\gamma_c$.

For non-perfect reflectivity, the transmission must be modified by a factor $\langle R^n \rangle$, which is defined by

$$\langle R^n \rangle = \int_0^{\theta_c} R^{n(\chi)}m(\chi)\,d\chi \Big/ \int_0^{\theta_c} m(\chi)\,d\chi. \tag{A4}$$

This may be evaluated exactly to give

$$\begin{aligned} \langle R^n \rangle &= \{(2\lambda_0^2/|\log R|)[1 - \exp(-|\log R|\lambda/\lambda_0)] + (\lambda_0 - \lambda)^2\}/(\lambda^2 + \lambda_0^2) \qquad \lambda < \lambda_0 \\ \langle R^n \rangle &= (|\log R|\lambda/\lambda_0)^{-1}[1 - \exp(-|\log R|\lambda/\lambda_0)] \qquad \lambda \geq \lambda_0. \end{aligned} \tag{A5}$$

In the limit of perfect reflectivity, both these expressions reduce to unity. In the limit of long wavelengths and/or large guide length with poor reflectivity, this expression becomes $\langle R^n \rangle = (|\log R|\lambda/\lambda_0)^{-1}$.

Also for the mean number of reflections we have

$$\langle n \rangle_R = \int_0^{\theta_c} R^{n(\chi)}n(\chi)m(\chi)\,d\chi \Big/ \int_0^{\theta_c} R^{n(\chi)}m(\chi)\,d\chi, \tag{A6}$$

which is evaluated to give

$$\begin{aligned} \langle n \rangle_R &= \{1 - [1 + |\log R|\lambda/\lambda_0]\exp(-|\log R|\lambda/\lambda_0)\} \\ &\quad \times \{|\log R|[1 - \exp(-|\log R|\lambda/\lambda_0) + |\log R|(\lambda_0 - \lambda)^2/2\lambda_0^2]\}^{-1} \qquad \lambda < \lambda_0 \\ \langle n \rangle_R &= \{1 - [1 + |\log R|\lambda/\lambda_0]\exp(-|\log R|\lambda/\lambda_0)\} \\ &\quad \times \{|\log R|[1 - \exp(-|\log R|\lambda/\lambda_0)]\}^{-1} \qquad \lambda \geq \lambda_0. \end{aligned} \tag{A7}$$

In the limit of perfect reflectivity, these expressions reduce to ($A3$). In the limit of long wavelengths and/or large guide length with poor reflectivity, $\langle n \rangle_R$ approaches $|\log R|^{-1}$ asymptotically.

Similarly for the mean square number of reflections we have

$$\langle n^2\rangle_R = \int_0^{\theta_c} R^{n(\chi)} n(\chi)^2 m(\chi)\, d\chi \Big/ \int_0^{\theta_c} R^{n(\chi)} m(\chi)\, d\chi, \quad (A8)$$

which is evaluated to give

$$\begin{aligned}\langle n^2\rangle_R = \{&2 - [2 + 2|\log R|\lambda/\lambda_0 + (|\log R|\lambda/\lambda_0)^2] \\ &\times \exp(-|\log R|\lambda/\lambda_0)\} \\ &\times \{|\log R|^2[1 - \exp(-|\log R|\lambda/\lambda_0) \\ &+ |\log R|(\lambda_0 - \lambda)^2/2\lambda_0^2]\}^{-1} \qquad \lambda < \lambda_0 \\ \langle n^2\rangle_R = \{&2 - [2 + 2|\log R|\lambda/\lambda_0 + (|\log R|\lambda/\lambda_0)^2] \\ &\times \exp(-|\log R|\lambda/\lambda_0)\} \\ &\times \{|\log R|^2[1 - \exp(-|\log R|\lambda/\lambda_0)]\}^{-1} \\ &\qquad \lambda \geq \lambda_0. \quad (A9)\end{aligned}$$

In the limit of perfect reflectivity, these expressions reduce to $2\lambda^3/3\lambda_0(\lambda^2 + \lambda_0^2)$ for $\lambda < \lambda_0$ and $1/3\ (\lambda/\lambda_0)^2$ for $\lambda \geq \lambda_0$, as expected. In the limit of long wavelengths and/or large guide length with poor reflectivity, $\langle n^2\rangle_R$ approaches $2|\log R|^{-2}$ asymptotically. Hence $\langle n^2\rangle_R - \langle n\rangle_R^2$ approaches $|\log R|^{-2}$ asymptotically.

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