Neutron lenses are used to focus neutron beams. They increase intensity on the sample and shrink the neutron spot size on the detector therefore reducing the minimum Q. The effects of focusing lenses on SANS resolution are discussed.

1. **FOCUSING LENSES’ BASIC EQUATIONS**

The focusing lenses’ basic equations are described here (Mildner et al, 2005; Hammouda-Mildner, 2007). The focal length for a set of N lenses of radius of curvature R and index of refraction n is given by:

\[ f = \frac{R}{2N(1 - n)} \]  \hspace{1cm} (1)

The index of refraction n is related to the material atomic density \( \rho \), neutron scattering length b, and neutron wavelength \( \lambda \) as:

\[ n = 1 - \frac{\rho b}{2\pi} \frac{\lambda^2}{\lambda} \]  \hspace{1cm} (2)

The focal length f is also related to the source-to-lenses distance \( L_1 \) and lenses-to-image distance \( L_4 \) as:

\[ \frac{1}{f} = \frac{1}{L_1} + \frac{1}{L_4} \]  \hspace{1cm} (3)

Combining the above two equations gives a relationship between the number N of lenses used and the neutron wavelength \( \lambda \) for an optimized instrument configuration where the detector is located at the focal spot.

\[ \frac{\pi}{\rho b c N \lambda^2} = \frac{L_1 L_4}{L_1 + L_4} \]  \hspace{1cm} (4)
Figure 1: Schematic representation of a focusing lens system showing an object (the neutron source aperture) and its image (on the detector plane). $L_1$ and $L_4$ are the source-to-sample and sample-to-detector distances and $f$ is the focal length. In practice, neutron focusing devices comprise many lenses used together.

For $\text{MgF}_2$ lenses the factor, one has:

$$\frac{\rho b}{\pi} = 1.632 \times 10^{-6} \text{Å}^{-2}$$

so that:

$$\frac{N\lambda^2}{R} \left( \frac{L_1 L_4}{L_1 + L_4} \right) = \frac{\pi}{\rho b} = 6.13 \times 10^5 \text{Å}^2.$$  

Consider lenses of radius of curvature $R = 2.5 \text{ cm}$ and height $H = 2.5 \text{ cm}$ that are thin at the center (1 mm thickness) in order to keep neutron transmission high. Source-to-sample and sample-to-image distances corresponding to the following SANS instrument configuration ($L_1 = 16.14 \text{ m}$, $L_4 = 13.19 \text{ m}$) give a focal length of

$$f = \left( \frac{L_1 L_4}{L_1 + L_4} \right) = 726 \text{ cm}.$$  

This gives $N\lambda^2 = 2111 \text{ Å}^2$. The use of 7 consecutive lenses ($N = 7$) focuses neutrons of wavelength $\lambda = 17.36 \text{ Å}$ with a focal distance of 726 cm. The use of 30 consecutive lenses focuses neutrons of wavelength $\lambda = 8.39 \text{ Å}$ down to the same focal spot. The use of 14 consecutive lenses corresponds to a focusing wavelength $\lambda = 12.20 \text{ Å}$.

For $\text{MgF}_2$, the index of refraction is:

$$n = 1 - 0.816 \times 10^{-6} \lambda^2.$$  

(7)
Note that the index of refraction of MgF₂ for neutrons is less than unity so that concave lenses focus neutrons whereas convex lenses defocus them. This is opposite to basic optics for light whereby the index of refraction is greater than unity.

2. RESOLUTION WITH FOCUSING LENSES

Consider a neutron beam with a triangular wavelength distribution and a focusing lens system optimized for the main wavelength \( \lambda_0 \) in that distribution. The main focal length is noted \( f_0 \) and corresponds to object-to-lens and lens-to-image distances of \( L_1 \) and \( L_2 \) respectively. Moreover, consider another wavelength \( \lambda \) within the same distribution and its corresponding focal length \( f \). The object-to-lens and lens-to-image distances are \( L_1 \) and \( L_4 \) respectively for this wavelength.

\[
\frac{1}{f_0} = \frac{1}{L_1} + \frac{1}{L_2} = \frac{2N}{R} \frac{\rho b}{2\pi} \lambda_0^2.
\]

\[
\frac{1}{f} = \frac{1}{L_1} + \frac{1}{L_4} = \frac{2N}{R} \frac{\rho b}{2\pi} \lambda^2.
\]

Figure 2: SANS focusing system showing the main image of the neutron source corresponding to the main neutron wavelength \( \lambda_0 \) and another image corresponding to another wavelength \( \lambda \).

In order to calculate the resolution with the lens system, the “geometry” contribution contains three terms: one that corresponds to the image of the source aperture; another
that corresponds to the sample aperture and a term that corresponds to averaging over a
detector cell.

\[
\begin{bmatrix}
\sigma^2_{x, \text{geo}} = \left(\frac{L_2}{L_4}\right)^2 \frac{R_4^2}{4} + \left(\frac{L_4 - L_2}{L_4}\right)^2 \frac{R_2^2}{4} + \frac{1}{3} \left(\frac{\Delta x_3}{2}\right)^2
\end{bmatrix}
\]  

(9)

Here \( R_4 \) is the radius of the image of the source aperture for the focal length \( f \) at
wavelength \( \lambda \). In order to see how the two scale factors were derived, consider the case
\( R_2 = 0 \) for which \( R_3 = \frac{L_2}{L_4} R_4 \), then the case of \( R_4 = 0 \) for which \( R_3 = \left(\frac{L_4 - L_2}{L_4}\right) R_2 \). \( \Delta x_3 \) is
the detector cell horizontal size. The image of the source aperture is given by \( R_4 = \frac{L_4}{L_1} R_1 \).

From the focusing equations, one obtains:
\[
\frac{1}{f_0} - \frac{1}{f} = \frac{1}{L_2} - \frac{1}{L_4} = \frac{L_4 - L_2}{L_2 L_4} = \frac{1}{f_0} \left( 1 - \left( \frac{\lambda}{\lambda_0} \right)^2 \right). \tag{10}
\]

Therefore:
\[
\sigma_{x_{geo}}^2 = \left( \frac{L_2}{L_1} \right)^2 \frac{R_1^2}{4} + \left( \frac{L_1 + L_2}{L_1} \right) \left( 1 - \left( \frac{\lambda}{\lambda_0} \right)^2 \right)^2 \frac{R_2^2}{4} + \frac{\Delta x_3^2}{12}. \tag{11}
\]

This is the result valid for any wavelength \( \lambda \). Around the focal wavelength \( \lambda_0 \), the averaging over the triangular wavelength distribution yields for the square term \( \left( 1 - \left( \frac{\lambda}{\lambda_0} \right)^2 \right)^2 \) the result of \( \frac{2}{3} \left( \frac{\Delta \lambda}{\lambda} \right)^2 \). Even though the subscript on \( \lambda_0 \) is dropped, it should be remembered that these results are valid only for the focusing wavelength.

Finally:
\[
\sigma_{y_{geo}}^2 = \left( \frac{L_2}{L_1} \right)^2 \frac{R_1^2}{4} + \left( \frac{L_1 + L_2}{L_1} \right) \left( \frac{\lambda_0}{\lambda} \right)^2 \left( \frac{\Delta \lambda}{\lambda} \right)^2 \frac{R_2^2}{4} + \frac{\Delta x_3^2}{12}. \tag{12}
\]

The spatial resolution in the vertical direction \( \sigma_y^2 \) involves the same terms as \( \sigma_x^2 \) along with contributions due to the gravity effect.

Neutrons follow a parabolic trajectory, which at the detector position (for \( z = L_1 + L_2 \)) is given by:
\[
y(L_1 + L_2) = y_0 - A \lambda^2 \quad \text{where} \quad A = L_2 (L_1 + L_2) \frac{g m^2}{2 h^2}. \tag{13}
\]

The effect of gravity and wavelength spread contribute terms of the following form to \( \sigma_y^2 \):
\[
[\sigma_{y_{grav}}^2] = < y(z)^2 > - < y(z) >^2 = A^2 [< \lambda^2 > - \lambda^2] = A^2 \lambda^4 \frac{2}{3} \left( \frac{\Delta \lambda}{\lambda} \right)^2. \tag{14}
\]

\[
[\sigma_{y_{wav}}^2] = Q^2 \frac{1}{6} \left( \frac{\Delta \lambda}{\lambda} \right)^2.
\]

In summary, the Q resolution is then obtained as:
\[ \sigma_{Qy}^2 = \left[ \sigma_{Qy}^2 \right]_{geo} + \left[ \sigma_{Qy}^2 \right]_{wav} + \left[ \sigma_{Qy}^2 \right]_{grav} \]  

\[ \sigma_{Qx}^2 = \left( \frac{2\pi}{\lambda L_2} \right)^2 \left[ \left( \frac{L_2}{L_1} \right)^2 \frac{R_2}{4} + \left( \frac{L_1 + L_2}{L_1} \right) \frac{2}{3} \frac{(\Delta \lambda)}{\lambda} \right] R_2^2 + \frac{1}{3} \left( \frac{(\Delta x_3)}{2} \right) + Q_x \frac{1}{6} \left( \frac{(\Delta \lambda)}{\lambda} \right)^2 \]

Using focusing lenses modifies the “sample” term only (second term proportional to \( R_2^2 \)). This term becomes much smaller with lenses. When lenses are used, the sample aperture \( R_2 \) can be made larger without much resolution penalty.

3. MINIMUM Q WITH FOCUSING LENSES

The minimum reachable value of Q starts at the edge of the beam spot. The geometry with focusing lenses is characterized by an umbra only (with no penumbra). The neutron beam spot at the detector is therefore characterized by a box (not a trapezoidal) profile. A simple optics argument gives for the edge of the beam umbra in the horizontal and vertical directions for each wavelength \( \lambda \) the following:

\[ X_{\text{min}}(\lambda) = \frac{L_2}{L_1} R_1 + \left( \frac{L_1 + L_2}{L_1} \right) \left[ 1 - \left( \frac{\lambda}{\lambda_0} \right)^2 \right] R_2 + \frac{\Delta x_3}{2} \]  

\[ Y_{\text{min}}(\lambda) = \frac{L_2}{L_1} R_1 + \left( \frac{L_1 + L_2}{L_1} \right) \left[ 1 - \left( \frac{\lambda}{\lambda_0} \right)^2 \right] R_2 + \frac{\Delta y_3}{2} + 2\lambda^2 \frac{\Delta \lambda}{\lambda} \]  

The last term in \( Y_{\text{min}}(\lambda) \) is due to gravity effect. Now the minimum achievable spot sizes are obtained by considering the part of the spot due to a wavelength spread \( \Delta \lambda \).

\[ X_{\text{min}}(\lambda) = \frac{L_2}{L_1} R_1 + \left( \frac{L_1 + L_2}{L_1} \right) \left[ 1 - \left( \frac{\lambda_0 + \Delta \lambda}{\lambda_0} \right)^2 \right] - \left( 1 - \left( \frac{\lambda_0}{\lambda_0} \right)^2 \right) R_2 + \frac{\Delta x_3}{2} \]  

The magnitude part reduces to:

\[ \left[ 1 - \left( \frac{\lambda_0 + \Delta \lambda}{\lambda_0} \right)^2 \right] - \left( 1 - \left( \frac{\lambda_0}{\lambda_0} \right)^2 \right) \approx 2 \left( \frac{\Delta \lambda}{\lambda} \right) \]
Now that the wavelength averaging has been performed, the $0$ subscript in $\lambda_0$ is dropped for simplicity. The horizontal and vertical beam spot sizes are:

$$X_{\text{min}} = \frac{L_2}{L_1} R_1 + L_1 + \frac{L_2}{L_1} 2 \left( \frac{\Delta \lambda}{\lambda} \right) R_2 + \frac{\Delta x_3}{2}$$

$$Y_{\text{min}} = \frac{L_2}{L_1} R_1 + L_1 + \frac{L_2}{L_1} 2 \left( \frac{\Delta \lambda}{\lambda} \right) R_2 + \frac{\Delta y_3}{2} + 2 \Delta \lambda^2 \left( \frac{\Delta \lambda}{\lambda} \right).$$

The corresponding values of the minimum $Q$ are:

$$Q_{\text{min}}^x = \frac{2\pi}{\lambda} \frac{X_{\text{min}}}{L_2}$$

$$Q_{\text{min}}^y = \frac{2\pi}{\lambda} \frac{Y_{\text{min}}}{L_2}$$

4. MEASURED SANS RESOLUTION

Specific Instrument Configuration

Consider the following instrument configuration:

$L_1 = 16.14$ m
$L_2 = 13.19$ m
$R_1 = 0.715$ cm
$R_2 = 0.635$ cm
$\Delta x_3 = \Delta y_3 = 0.5$ cm
$\frac{\Delta \lambda}{\lambda} = 0.13.$

This gives $A = 0.01189$ cm$^2$/Å$^2$.

Measurements with Focusing Lenses

Neutron optics measurements were made using a set of 7 consecutive biconcave MgF$_2$ lenses (described in a previous chapter) inserted in the beam just before the sample aperture. This set corresponds to a focal wavelength $\lambda_0$ around 17.36 Å.

The measured position of the neutron beam spot on the detector agrees with predictions.
Figure 4: Variation of the neutron beam spot positions with wavelength.

The beam spot resolution has strong (parabolic) wavelength dependence both in the x and in the y directions. The minimum resolution in the horizontal direction corresponds to a focal wavelength $\lambda_0$. The minimum in the x direction ($\lambda_0 = 17.2 \text{ Å}$) is taken to be the focal wavelength for our focusing arrangement since the x direction is independent of gravity effects. A procedure of using slice cuts across the beam spot was used to obtain these plotted results (including the $\sqrt{1.45}$ scaling discussed in a previous chapter). The calculated trends agree with the measured ones.
With Neutron Lenses

Figure 5: Variation of the spot size standard deviation with wavelength in the horizontal and vertical directions.

Variation of the minimum spot sizes as a function of increasing wavelength is characterized by a minimum around $\lambda_0 = 17.2$ Å. The measured values have been chosen conservatively and are found to be overestimates that are higher than the calculated values.
Discussion

The use of converging lenses has the advantage of allowing the opening up of the sample aperture (i.e., increasing $R_2$) without penalty in resolution. This happens because the penumbra is minimized when lenses are used. The main effect is increased neutron current on sample.

Refractive lenses are characterized by chromatic aberrations that show up as a dependence of both the variance $\sigma_x^2$ and $X_{\text{min}}$ on $(\Delta \lambda / \lambda)$. In order to reduce these chromatic aberrations, $(\Delta \lambda / \lambda)$ could be made smaller; which would result in a penalty in neutron current on sample. Focusing devices that use reflection (rather then refraction) optics (such as elliptical or torroidal mirrors) are not hampered by such chromatic aberrations.
5. LENS TRANSMISSION

The transmission of a set of 7 concave spherical lenses is calculated and compared to transmission measurements. Consider a lens of spherical radius $R$ and thickness $2h$ at the center and assume that the beam defining aperture has a radius of $B$.

![Schematics of the lens geometry.](image)

The transmission of one focusing lens averaged over the beam aperture is given by:

$$T_i = \frac{1}{\pi B^2} \int_0^B dy 2\pi y \exp[-2\Sigma_i (h + R - z)]$$  \hspace{1cm} (22)

Here $y$ is the vertical coordinate, $z$ is the horizontal coordinate obeying $z = \sqrt{R^2 - y^2}$ and $\Sigma_i$ is the macroscopic cross section for MgF$_2$. Note that $\Sigma_i$ varies with neutron wavelength as $\Sigma_i = 0.000513 \lambda$ where $\lambda$ is in Å and $\Sigma_i$ in mm$^{-1}$. This variation was measured using a uniform thickness slab of MgF$_2$.

Performing the simple integration, one obtains:

$$T_i = \frac{\exp[-2\Sigma_i (h + R)]}{2(\Sigma_i B)^2} \left\{ [1 - 2\Sigma_i \sqrt{R^2 - B^2}] \exp[2\Sigma_i \sqrt{R^2 - B^2}] - (1 - 2\Sigma_i) \Sigma_i \right\}$$  \hspace{1cm} (23)

The transmission of a set of 7 focusing lenses is given by $T_7 = T_1^7$.

The calculated and measured transmissions for the 7-lens system are compared for increasing neutron wavelength.
Figure 8: Calculated and measured neutron transmissions for a 7-lens system.

The calculated and measured transmissions agree only partially.

REFERENCES


QUESTIONS

1. What is the main difference between focusing lenses for neutrons and focusing lenses for light?
2. Name a typical neutron focusing lens material.
3. When using neutron focusing lenses, what term of the instrumental resolution variance is modified? What is the advantage of this?
4. What are chromatic aberrations?
5. Do reflective optical devices suffer from chromatic aberrations? Name a refractive optics focusing device.

6. Given the transmission $T_1$ of one focusing lens, calculate the transmission $T_7$ of a 7-lens system.

7. Using many lens systems, could one build a neutron microscope?

8. What are the two main figures of merit for making a good refractive material to be used for making neutron lenses?

**ANSWERS**

1. **Focusing lenses for neutrons are concave. Focusing lenses for light are convex.** This is due to the fact that the index of refraction for neutrons is less than one while that for light is greater than one for most typical focusing materials. This is due to the fact that the scattering length for most materials is positive. Exceptions include hydrogen which has a negative scattering length.

2. **MgF₂** is a commonly used neutron focusing lens material.

3. **The use of focusing lenses modifies the “sample aperture” term of the resolution variance.** This term becomes much smaller even for larger source apertures. The advantage is a larger neutron current on sample.

4. Chromatic aberrations correspond to the de-focusing effect for different wavelengths. The position of the source aperture image changes with wavelength thereby “blurring” the “image”.

5. **There are no chromatic aberrations with refractive optics.** Torroidal or elliptical mirrors are typical refractive optics focusing devices.

6. The transmission of a 7-lens system is given by $T_7 = T_1^7$ where $T_1$ is the transmission of one lens.

7. If one had lenses after the sample, one could obtain magnification using a neutron beam (neutron microscope). Given the low neutron wavelengths $\lambda$ (compared to light) the focal length $f$ is very long ($f = \pi R/N \rho b \lambda^2$). Chromatic aberrations, the required long flight paths and coarse detector resolution give only modest magnification and a fuzzy picture. Note that the magnification factor can be worked out to be $M = \frac{f}{L_1 - f}$ where $f$ is the focal length and $L_1$ is the object (sample in this case)-to-lenses distance. Note that $L_1 = f$ would yield high magnification. However, this condition would require that the lenses-to-image distance $L_4$ be infinite (recall that $1/L_4 = 1/f - 1/L_1$). This is not realistic.

8. The two figures of merit for refractive materials for making neutron lenses are as follows. (1) High density $\rho$ and high coherent scattering length $b$ in order to make the index of refraction $n$ as small as possible. Recall that $1 - n = \frac{\rho b}{2 \pi} \lambda^2$. Making $1-n$ large (i.e., $n$ small) reduces the focal distance $f$ since $f = R/2N(1-n)$ where $R$ is the lens radius and $N$ is the number of lenses. (2) One would want to minimize the incoherent and absorption scattering cross sections $\Sigma_i$ and $\Sigma_a$ in order to minimize background and maximize lens transmission.